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5d Black holes, wrapped fivebranes and 3d Chern-Simons Super Yang-Mills

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Abstract: We study extremal and non-extremal generalizations of the regular non-abelian solution found by Chamseddine and Volkov in 5d N=4 gauged supergravity, which has been shown by Maldacena and Nastase to describe a system of NS5-branes wrapping an S^3 dual to three-dimensional $\mathrm{U}(N)$ $\mathcal{N}=1$ supersymmetric Yang-Mills with Chern-Simons coupling $k = N/2$. All black hole solutions have a temperature larger than the Hagedorn temperature T_c of the little string theory and their entropy decreases as the temperature increases. This is a sign that the system is thermodynamically unstable above T_c. We have also found an analytical solution describing NS5-branes wrapped on a constant radius S^3 and involving a linear dilaton. Its non-extremal generalization has a temperature equal to $2T_c$.

Keywords: Black Holes in String Theory, AdS-CFT and dS-CFT Correspondence.

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1. Introduction

In [1], Witten studied three-dimensional U(N) $\mathcal{N}=1$ Super Yang-Mills with Chern-Simons coupling k. He showed that this model preserves supersymmetry for $k \geq N/2$, and he conjectured that supersymmetry is spontaneously broken for $k < N/2$. In the limit case, $k = N/2$, the Witten index is one, so there is one ground state and furthermore this vacuum is confining.

Maldacena and Nastase found a supergravity dual of this model [2]. In particular, they showed that the lifting to IIB supergravity of the regular solution found by Chamseddine and Volkov in [3], in the context of 5d N=4 gauged supergravity, corresponds to a system of NS5-branes wrapping a three-sphere S^3 with a twisting that preserves two supercharges. This S^3 is contractible and the low-energy limit of this system exactly reproduces the model above.

This solution has a non-trivial flux of the NS three-form H on the wrapped S^3, which gives rise to a Chern-Simons coupling equal to $N/2$. If we add extra fivebranes wrapping the three-sphere transverse to the original set of N branes, we increase the flux of H and therefore increase the Chern-Simons coupling k. Conversely, adding antibranes reduces the value of k, but this breaks supersymmetry as conjectured by Witten [2].

Brane configurations realizing three-dimensional Super Yang-Mills theories with Chern-Simons couplings were constructed in [4]. Furthermore, a derivation of the supersymmetry breaking conditions for these models using the s-rule was given in [5] and [6].

The aim of this paper is to study this three-dimensional model at finite temperature and the transition between the confined and the deconfined phase in the spirit of [7]. Our analysis will essentially follow the lines of [8], where the case of 4d pure $\mathcal{N} = 1$ Super Yang-Mills was investigated by considering non-extremal deformations of the supergravity dual proposed by Maldacena and Nuñez in [9].

2. The supergravity description

We briefly review the system studied in [2]. Let us consider a system of type IIB NS5-branes wrapped on a three-sphere S^3. The worldvolume theory will not be supersymmetric unless there is a proper twisting [10]. This means that the spin connection of the curved part of the worldvolume, namely S^3, has to be embedded into the R-symmetry group of the theory, which is also the structure group of the normal bundle of the NS5-branes, namely $\mathrm{SO}(4) \cong \mathrm{SU}(2)_L \times \mathrm{SU}(2)_R$. Since the tangent space of S^3 is three-dimensional, the spin connection lies in SU(2). Therefore, the twisting amounts to choosing an embedding of SU(2) into SO(4). In particular, setting the spin connection to be in $\mathrm{SU}(2)_L$ preserves $\mathcal{N} = 1$ supersymmetry in three dimensions.

If the sphere is large, then at low energies compared to the six-dimensional coupling constant there is a six-dimensional $\mathrm{U}(N)$ theory on the worldvolume of N NS5-branes. The theory will be effectively three-dimensional at energies lower than the inverse radius of the sphere and its coupling will be weak if $Vol(S^3) \gg (\alpha')^{3/2}$. The only massless fields are the gauge bosons and the gauginos.

In the regime where the supergravity description is valid, the scale where the 3d theory becomes strongly coupled and the Kaluza-Klein scale have the same order of magnitude.

An important twist in the story is that a flux of the NS three-form field strength H on the wrapped sphere induces a Chern-Simons coupling in three dimensions [11]. In the S-dual description, namely where we consider $D5$-branes wrapping S^3, this is a consequence of the Wess-Zumino coupling between the RR two-form C and the worldvolume gauge field strength [12]

$$\begin{aligned}\frac{1}{16\pi^3}\int_{\Sigma_6} C\wedge \mathrm{Tr}(F\wedge F) &= -\frac{1}{16\pi^3}\int_{\Sigma_6} G\wedge \mathrm{Tr}\left(A\wedge dA+\frac{2}{3}A^3\right)\\ &= -\frac{k_6}{4\pi}\int_{\Sigma_3}\mathrm{Tr}\left(A\wedge dA+\frac{2}{3}A^3\right),\end{aligned} \tag{2.1}$$

where k_6 is the Chern-Simons coupling appearing in the six-dimensional lagrangian. It is important to stress that the effective three-dimensional Chern-Simons coupling may in principle be different from k_6. In general, integrating out massive fermions induces a shift of this coupling whose sign depends on the sign of the fermion mass. In [2], the authors showed that the final coupling is given by

$$k = k_6 - \frac{N}{2}\,. \tag{2.2}$$

2.1 The dual gravity solution

As explained in [2], following the approach pioneered in [13] and exploited in [9] to provide a supergravity dual of pure $\mathcal{N}=1$ SYM in four dimensions, the natural setup to look for such a gravity solution dual to the above SYM-CS theory would be minimal 7d gauged supergravity with gauge group SU(2) [14]. This theory contains the metric, a dilaton ϕ, the SU(2) gauge fields A^i, $i=1,2,3$, and a three-form field strength h. A solution of this theory can then be uplifted to type IIB using the formulas in [15, 16, 17].

The seven dimensions manage to accomodate the six-dimensional brane worldvolume, comprising the three-dimensional flat part and the three-sphere, and a radial direction. Most importantly, the SU(2) gauge fields describe the $\mathrm{SU}(2)_L$ within the R-symmetry group of the NS5-branes. We can choose the large radius asymptotics of the solution to be

$$\begin{aligned} ds^2_{7,str} &\sim dx^2_{2+1} + N\alpha'\left[dr^2 + R^2(r)d\Omega_3^2\right], \\ \frac{1}{(2\pi)^2}\int_{S^3_\infty} h &= k\,, \\ A^i &\sim \frac{1}{2}\theta^i\,, \\ \phi &\sim -r\,, \end{aligned} \tag{2.3}$$

where the θ^i's are the left-invariant one-forms on the three-sphere satisfying $d\theta^i + (1/2)\times \epsilon_{ijk}\theta^j\wedge\theta^k = 0$. The radius of S^3 will have a non-trivial r dependence. In particular, $\mathbb{R}^2$ will vanish in the $r\to 0$ limit whereas $\mathbb{R}^2 \sim r$ for large r.

A solution of this SU(2) seven-dimensional gauged supergravity can be mapped to a solution of type IIB supergravity where we keep only a subset of the bosonic fields, in particular the metric, the dilaton and the NS three-form field strength H [15, 16, 17]

$$\begin{aligned} ds^2_{10,str} &= ds^2_{7,str} + N\alpha'\frac{1}{4}\sum_{i=1}^{3}(\hat\theta^i - A^i)^2 \\ H &= N\left[-\frac{1}{24}\epsilon_{ijk}(\hat\theta^i - A^i)(\hat\theta^j - A^j)(\hat\theta^k - A^k) + \frac{1}{4}F^i(\hat\theta^i - A^i)\right] + h\,, \end{aligned} \tag{2.4}$$

where A^i and h are the seven-dimensional gauge fields and three-form respectively. Note that the ten-dimensional dilaton is the same as the seven-dimensional one. By $\hat\theta^i$'s we denote the left-invariant one-forms on the three-sphere transverse to the branes. Note that the transverse three-sphere is not contractible.

Solutions to seven-dimensional supergravity describing this system of wrapped 5-branes had been considered in [11]. However, the solutions studied there develop a "bad" singularity at the origin, according to the criterion of [18].

It was realized in [19] and [2] that a regular solution could be achieved by uplifting the non-singular BPS solution found by Chamseddine and Volkov in five dimensions [3]. These authors considered a proper truncation of the five-dimensional N=4 SU(2) × U(1) supergravity introduced by Romans in [20]. The bosonic fields are the metric, the SU(2) gauge fields A^i, a U(1) gauge field a, with field strength $\mathcal{F} = da$, a pair of two-forms and the dilaton. Both these two-forms and the abelian coupling can be consistently set to zero on-shell.

Chamseddine and Volkov started with the following ansatz

$$ds^2_{5,str} = -dt^2 + N\alpha' \left[dr^2 + R(r)^2 d\Omega_3^2\right] \tag{2.5}$$

$$A^i = \frac{w(r)+1}{2}\theta^i \tag{2.6}$$

$$\mathcal{F} = Q(r)\, dtdr\,, \tag{2.7}$$

and found two BPS solutions, both of them preserving two supercharges. The details of these solutions will be rederived and discussed later.

As we said, the solution can be uplifted to seven-dimensional gauged supergravity using formulas in [21, 22]. In particular

$$ds^2_{7,str} = ds^2_{5,str} + dx_1^2 + dx_2^2\,,$$

and the SU(2) gauge fields are the same. The abelian field strength $\mathcal{F}$ gives rise to a four-form field strength $F_{(4)} = \mathcal{F} dx_1 dx_2$ which can be finally dualized to yield $h = e^{2\phi} *_{7,str} F_{(4)}$ [19, 2].

Supergravity duals of three-dimensional theories with $\mathcal{N}=1$ and $\mathcal{N}=2$ supersymmetry were also constructed in [23]–[26]. Solutions of eleven-dimensional supergravity corresponding to RG flows between 3d theories with $\mathcal{N}=1$, $\mathcal{N}=2$ and $\mathcal{N}=3$ supersymmetry were given in [27]. RG flows to 3d theories were also studied in [28] in the context of 6d gauged supergravity.

In summary, the supersymmetric solution has the form $\mathbb{R}^{1,2} \times M_7$ with non-trivial NS flux and dilaton and it preserves two supercharges. Note that the manifold M_7 is not Ricci flat and thus cannot be a G_2-manifold. However, the analysis of [29] shows that M_7 is actually endowed with a G_2-structure.

3. The type IIB supergravity description of NS-branes on S^3

We shall study solutions in the following subsector of the type IIB supergravity action

$$S_{10} = \frac{1}{4}\int d^{10}x\sqrt{-g}\left(R - \frac{1}{2}(\partial\Phi)^2 - \frac{1}{12}e^{-\Phi}H^2\right), \tag{3.1}$$

where $H = dB = (1/6)H_{MNS}dx^M \wedge dx^N \wedge dx^S$ is the NS field strength and Φ is the dilaton. Motivated by the previous discussion on the uplifting of the five-dimensional solutions, we will start from the following ansatz for the ten-dimensional string frame metric

$$\begin{aligned} ds^2_{str,10} &= -e^{2X(r)}dt^2 + N\alpha'\left[e^{2Y(r)-2X(r)}dr^2 + R^2 d\Omega_3^2\right] + \\ &+ \sum_{i=1}^{2} dx^i dx^i + \frac{N\alpha'}{4}\sum_{i=1}^{3}(\hat{\theta}^i - A^i)^2 \,, \end{aligned} \tag{3.2}$$

where $d\Omega_3^2 = \sum_i \theta_i^2$ and the gauge fields A^i are given by (2.6). The Einstein metric is given by $ds^2_E = e^{-\Phi/2}ds^2_{str}$. The ansatz for the NS three-form field strength is given by (2.4) where

$$h = Nf(r)\frac{1}{6}\epsilon_{ijk}\theta^i\theta^j\theta^k \,, \tag{3.3}$$

and

$$F^i = dA^i + \frac{1}{2}\epsilon_{ijk}A^j \wedge A^k = \frac{1}{2}w' dr \wedge \theta^i + \frac{w^2-1}{8}\epsilon_{ijk}\theta^j \wedge \theta^k \,. \tag{3.4}$$

The function $f(r)$ is determined by 5d supergravity to be

$$f(r) = \frac{w^3 - 3w + 4\kappa}{16} \,. \tag{3.5}$$

Note that the NS three-form H is closed, which, as explained in [29], is one of the conditions M_7 has to satisfy for the existence of supersymmetric solutions. The constant κ is related to the Chern-Simons parameter k and the number of colors N. In fact, by (2.1)(2.2)

$$k_6 = \frac{1}{4\pi^2}\int_{S^3_\infty} H = \frac{N}{2} + N\kappa = \frac{N}{2} + k \,, \tag{3.6}$$

where we used the fact that $\lim_{r\to\infty} w(r) = 0$. This implies that

$$\kappa = \frac{k}{N} \,. \tag{3.7}$$

We will see that the regular BPS solution corresponds to $\kappa = 1/2$ and hence to $k = \frac{N}{2}$ [2]. Inserting the ansatz (3.2)–(3.5) into the type IIB action (3.1), integrating and dropping a surface term and an overall constant factor, we find $S_{eff} = \int dt \int dr L$, where

$$\begin{aligned} L &= \frac{1}{256}e^{-2\Phi+2X-Y}R\left(16R^2\Phi'^2 - 48R\Phi' R' + 24R'^2 - 3w'^2 - 16R^2\Phi' X' + 24RR'X'\right) - \\ &- \frac{1}{256R^3}e^{-2\Phi+Y}\left(128f^2 + 3R^2(w^2-1)^2 - 24R^4 - 16R^6\right) \,, \end{aligned} \tag{3.8}$$

and primes denote derivatives with respect to r. Note that Y plays the role of a lagrangian multiplier. It enforces a constraint that is a remnant of the residual invariance of the ansatz under a reparametrization of the radial coordinate. Varying the above one-dimensional lagrangian yields the following system of equations

$$\Phi'' = \frac{3w'^2}{8R^2} + \frac{3R''}{2R}\,, \tag{3.9}$$

$$w'' + (\frac{\nu'}{\nu} + \frac{R'}{R} - 2\Phi')w' - \frac{(w^2-1)(4\kappa + (4R^2-3)w + w^3)}{2\nu R^4} = 0\,, \tag{3.10}$$

$$R'' + \frac{w'^2 - 4R'^2}{R} + \frac{4(R^2+1)}{\nu R} + \frac{2\nu'}{\nu}(R\Phi' - R') - 4R\Phi'^2 + 10R'\Phi' - \frac{(w^2-1)^2}{4\nu R^3} = 0\,, \tag{3.11}$$

$$\nu' = K\frac{e^{2\Phi}}{R^3}\,, \tag{3.12}$$

where $\nu \equiv e^{2X}$, K is a constant and we set $Y = 0$. This is supplemented by the constraint

$$\frac{3+2R^2}{32} - 3\frac{(w^2-1)^2}{256R^2} - \frac{(w^3-3w+4\kappa)^2}{512R^4} - \frac{1}{16}\nu R^2\Phi'^2 + \frac{3}{16}\nu\Phi' RR' - \\ -\frac{3}{32}\nu R'^2 + \frac{3}{256}\nu w'^2 + \frac{1}{32}\nu' R^2\Phi' - \frac{3}{64}\nu' RR' = 0\,. \tag{3.13}$$

The above system is invariant under $\Phi \to \Phi + C$, $K \to e^{-2C}K$ and separately under $w \to -w$, $\kappa \to -\kappa$. Furthermore, the system is symmetric under a constant rescaling of the radial coordinate. In fact, if $\{\Phi(r), R(r), w(r), \nu(r)\}$ is a solution, then $\{\Phi(e^{2d}r) - d, R(e^{2d}r), w(e^{2d}r), e^{-4d}\nu(e^{2d}r)\}$ is a solution as well. The same holds for translations, by replacing r with $r + r_0$ in the argument of each function.

4. Extremal solutions

We can now study solutions of this system. We can consider two distinct cases: $K = 0$ or $K \neq 0$. The first is the extremal case, where the solution has SO(1,2) symmetry. This corresponds to $\nu = e^{2X} = const.$ By virtue of the rescaling symmetry, we can always set $\nu = 1$. Then the ten-dimensional string frame metric reads

$$ds^2_{str,10} = -dt^2 + N\alpha'\left[dr^2 + R^2 d\Omega_3^2\right] + \sum_{i=1}^{2} dx^i dx^i + \frac{N\alpha'}{4}\sum_{i=1}^{3}(\hat{\theta}^i - A^i)^2\,. \tag{4.1}$$

Thus, we see that the (t, r) part of the metric is flat and the solutions will be either globally regular or have naked singularities.

We will also consider the non-extremal case, $K \neq 0$. In fact we want to find black hole solutions with a regular event horizon, since these correspond to the finite temperature gauge theory in a deconfined phase [7].

4.1 BPS solutions

The goal of this section is to find the supersymmetric solutions of the above system. They correspond to the case $K = 0$ and will actually satisfy a system of first order equations, which was first derived in the context of $D = 5$, $\mathcal{N} = 4$ gauged supergravity in [3]. We will

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not follow the canonical approach of setting to zero the fermion supersymmetry variations, but instead we will try to find a superpotential W for the action. The first order system relevant to this ansatz was also calculated by [2]. The effective lagrangian with $K = 0$

$$
\begin{aligned}
L = & \frac{e^{-2\Phi-Y}}{256} R\left(16R^2\Phi'^2 - 48R\Phi'R' + 24R'^2 - 3w'^2\right) - \\
& - \frac{e^{-2\Phi+Y}}{512R^3}\left[(w^3 - 3w + 4\kappa)^2 + 6(w^2-1)^2R^2 - 48R^4 - 32R^6\right]
\end{aligned} \tag{4.2}
$$

can be rewritten in the form

$$
L = G_{ij}(y)\frac{dy^i}{dr}\frac{dy^j}{dr} - U(y)\,, \qquad y^i = (s, g, w)\,,
$$

where $s = \Phi - 3/2g$, $R = e^g$, and the potential U and the metric G_{ij} are

$$
U = \frac{e^{-2s-6g+Y}}{512}\left[(w^3 - 3w + 4\kappa)^2 + 6(w^2-1)^2e^{2g} - 48e^{4g} - 32e^{6g}\right], \tag{4.3}
$$

$$
G_{ij} = \frac{e^{-2s-Y}}{16}\,\mathrm{diag}\left(1, -\frac{3}{4}, -\frac{3}{16}e^{-2g}\right). \tag{4.4}
$$

A direct calculation shows that the potential U can be represented as

$$
U = -G^{ij}\frac{\partial W}{\partial y^i}\frac{\partial W}{\partial y^j}\,,
$$

where the superpotential W reads

$$
W = \pm\frac{3}{64}e^{-g-2s}\sqrt{M}\,, \tag{4.5}
$$

with

$$
M = \frac{4e^{2g}}{9} - \frac{2}{3}(w^2-1) + \frac{1}{4}e^{-2g}(w^2-1)^2 + \left[\frac{e^{-2g}}{24}(2w^3 - 6w + 8\kappa) - w\right]^2. \tag{4.6}
$$

Thus the lagrangian is equivalent to

$$
L = G_{ij}\left(\frac{dy^i}{dr} - G^{ik}\frac{\partial W}{\partial y^k}\right)\left(\frac{dy^j}{dr} - G^{jl}\frac{\partial W}{\partial y^l}\right) + 2W'\,,
$$

which implies that the solutions to the first order equations

$$
\frac{dy^i}{dr} = G^{ik}\frac{\partial W}{\partial y^k}\,,
$$

solve the second order system as well. We obtain

$$
\frac{dR}{dr} = \frac{1}{6\sqrt{M}}\left[\frac{(w^3 - 3w + 4\kappa)^2}{8R^4} + \frac{(w^4 - 8\kappa w + 3)}{R^2} + 2(w^2+2)\right], \tag{4.7}
$$

$$
\frac{dw}{dr} = \frac{4R}{3\sqrt{M}}\left[\frac{1}{16R^4}(w^3 - 3w + 4\kappa)(1-w^2) + \frac{1}{2R^2}(2\kappa - w^3) - w\right], \tag{4.8}
$$

$$
\frac{d\Phi}{dr} = \frac{3}{2R}\frac{dR}{dr} - \frac{3}{2}\frac{\sqrt{M}}{R}. \tag{4.9}
$$

If $\kappa = 0$, $w \equiv 0$ solves the system and we retrieve the singular BPS solution found in [3]

$$R = \sqrt{2r}\,, \qquad \Phi = \Phi_0 - r + \frac{3}{8}\log r\,. \tag{4.10}$$

Note that, by (3.5), the abelian gauge field is trivial. On the other hand, the supersymmetric solution which is dual to the $D = 3$ $\mathrm{U}(N)$ $\mathcal{N} = 1$ super Chern-Simons theory with $k = N/2$ corresponds to $\kappa = 1/2$ [3]. We can set the radius R to be vanishing at $r = 0$. We will see in the next section how the requirement of regularity of the solution essentially fixes κ. The fields have the following asymptotic behaviour for small r

$$\begin{aligned} w &= 1 - \frac{1}{3}r^2 + \mathcal{O}(r^4)\,, \qquad R = r - \frac{1}{12}r^3 + \mathcal{O}(r^5)\,, \\ \Phi &= \Phi_0 - \frac{7}{24}r^2 + \mathcal{O}(r^4)\,, \end{aligned} \tag{4.11}$$

whereas for large r

$$\begin{aligned} R &= \sqrt{2r} + \mathcal{O}(e^{-2r})\,, \qquad \Phi = \Phi_0 - r + \frac{3}{8}\log r + \mathcal{O}(e^{-2r})\,, \\ w &= \mathcal{O}(e^{-2r})\,. \end{aligned} \tag{4.12}$$

Note that the abelian gauge field is non-trivial in this case.

4.2 Non-BPS solutions

Let us proceed to find extremal solutions of the second order system which are not supersymmetric.

4.2.1 Special solutions: linear dilaton & constant radius

There are special solutions characterized by a constant radius and a linear dilaton. They are given by

$$(w,\kappa) = (0,0)\,, \qquad R = \frac{1}{2}\,, \qquad \Phi = \Phi_0 - Zr\,, \qquad \nu = \frac{4}{Z^2}\,. \tag{4.13}$$

Setting $Z = 2$, the string frame metric reads

$$ds^2_{10,str} = -dt^2 + \sum_{n=1}^{2} dx_n dx_n + N\alpha' \left[dr^2 + dM_6^2\right]\,, \tag{4.14}$$

where

$$dM_6^2 = \frac{1}{4}\sum_{i=1}^{3}(\theta^i)^2 + \frac{1}{4}\sum_{i=1}^{3}\left(\hat\theta^i - \frac{1}{2}\theta^i\right)^2\,, \tag{4.15}$$

is a homogeneous metric on $S^3 \times S^3$. The three-form is given by

$$H = N\left[-\frac{1}{24}\epsilon_{ijk}\left(\hat\theta^i - \frac{1}{2}\theta^i\right)\left(\hat\theta^j - \frac{1}{2}\theta^j\right)\left(\hat\theta^k - \frac{1}{2}\theta^k\right) - \frac{1}{32}\epsilon_{ijk}\theta^i\theta^j\left(\hat\theta^k - \frac{1}{2}\theta^k\right)\right]. \tag{4.16}$$

We expect the above background to be described by a WZW model, probably a deformation[1] of SU(2) × SU(2). A similar solution, corresponding to a system of NS5-branes wrapping a constant radius S^2, and the related WZW model were found in [8]. The description is in terms of an SU(2) × SU(2)/ U(1) coset model studied in [30] and based on a general construction by [31].

We will also see that (4.13) have a simple non-extremal generalization. It can also be checked that there are no solutions with constant radius and linear dilaton for $(w, \kappa) = (1, \frac{1}{2})$.

4.2.2 Globally regular solutions

In this subsection, we will consider general extremal non-BPS solutions of the second order system with a non-constant w. For $K = 0$, $\nu = 1$, the system reduces to

$$\Phi'' = \frac{3w'^2}{8R^2} + \frac{3R''}{2R}, \tag{4.17}$$

$$w'' + \left(\frac{R'}{R} - 2\Phi'\right) w' - \frac{(w^2-1)(4\kappa + (4R^2-3)w + w^3)}{2R^4} = 0\,, \tag{4.18}$$

$$R'' + \frac{w'^2 - 4R'^2}{R} + \frac{4(R^2+1)}{R} - 4R\Phi'^2 + 10R'\Phi' - \frac{(w^2-1)^2}{4R^3} = 0\,, \tag{4.19}$$

$$\begin{aligned}\frac{3+2R^2}{32} - 3\frac{(w^2-1)^2}{256R^2} - \frac{(w^3-3w+4\kappa)^2}{512R^4} - \frac{1}{16}R^2\Phi'^2 + \frac{3}{16}\Phi' R R' - \\ - \frac{3}{32}R'^2 + \frac{3}{256}w'^2 = 0\,.\end{aligned} \tag{4.20}$$

We are interested in globally regular solutions only, for which spacetime is geodesically complete. In particular, we will consider solutions with a regular origin, which is the point r_0 where the three-sphere radius R vanishes but the curvature is bounded. Note that, by the above equations, this condition implies that $\kappa = 1/2$ modulo the $w \to -w$, $\kappa \to -\kappa$ symmetry.

By translational symmetry, we can set $r_0 = 0$. We cannot analytically continue the manifold to negative r and so we can assume $r \geq 0$. Then, the system admits a one-parameter family of solutions with the following small r Taylor expansion

$$\begin{aligned} w &= 1 - br^2 + \mathcal{O}(r^4)\,, \qquad R = r - \frac{2+9b^2}{36}r^3 + \mathcal{O}(r^5)\,, \\ \Phi &= \Phi_0 - \frac{2+3b^2}{8}r^2 + \mathcal{O}(r^4)\,. \end{aligned} \tag{4.21}$$

We see that b and Φ_0 are free parameters. The value

$$b = \frac{1}{3}\,,$$

corresponds to the regular BPS solution. In order to find the regular non-BPS deformations, we will numerically integrate the system (4.17)–(4.20), using (4.21) as the boundary conditions at $r = 0$.

[1] We thank J. Maldacena for this suggestion.

Numerically we find that the allowed range for b is $[0,1[$. This is due to the fact that for $b \geq 1$, R goes to zero again at a finite value of r. For $0 < b < 1/3$, the function w is always positive, whereas for $b > 1/3$, it has one node. In both cases $\lim_{r\to\infty} w = 0$. The regular BPS solution corresponds to $b = 1/3$, in which case w goes to zero exponentially.

4.3 Asymptotic behaviour of the solutions

In order to evaluate the energy and free energy of a solution, we will need to know explicitly its asymptotic behaviour in the limit of large r.

In the following, we will treat both the extremal and the non-extremal cases at the same time and assume that for large r the radius R is not bounded. With this assumption, we find the following

$$\begin{aligned} R &= \sqrt{2x} - \left(\frac{\gamma^2}{4\sqrt{2}x^{3/2}} + \cdots\right) + \sqrt{2}\mathcal{P}x^{3/4}e^{-2x}\left(1 + \frac{1}{x} + \cdots\right) + \mathcal{O}(e^{-3x})\,, \\ \Phi &= \Phi_\infty - x + \frac{3}{8}\log x - \left(\frac{15\gamma^2}{128x^2} + \cdots\right) \\ &\quad + \frac{3}{2}\mathcal{P}x^{1/4}e^{-2x}\left(1 + \frac{1}{2x} + \cdots\right) + \mathcal{O}(e^{-3x})\,, \\ w &= \frac{\gamma}{\sqrt{x}}(1 + \cdots) + \mathcal{C}x^{1/2}e^{-2x}(1 + \cdots) + \mathcal{O}(e^{-3x})\,, \\ \nu &= \frac{1}{\mu^2}\left(1 - \frac{K}{2^{5/2}\,x^{3/4}}e^{-2x+2\Phi_\infty}(1 + \cdots)\right)\,, \qquad x \equiv \mu(r + r_\infty)\,, \end{aligned} \tag{4.22}$$

where μ, r_∞, $\mathcal{P}$, Φ_∞, γ and $\mathcal{C}$ are integration constants. Note that there is 6 of them, due to the fact that (3.9)–(3.13) can be viewed as a system of 7 first order equations supplemented by one constraint, which appeared due to the remaining reparametrization invariance of the ansatz. The system allows for solutions with bounded R as well. The parameter μ naturally appears due to the scaling symmetry of the system. The BPS solution has $\gamma_{BPS} = 0$, since w vanishes exponentially, and $\mathcal{P}_{BPS} = 0$.

5. The non-extremal case: black hole solutions

5.1 Solutions with a regular horizon

In this section, we are going to study non-extremal solutions, where the function ν is not constant, corresponding to the parameter K being non-vanishing. The ten-dimensional string frame metric reads

$$ds^2_{str,10} = -\nu dt^2 + N\alpha'\left[\frac{1}{\nu}dr^2 + R^2 d\Omega_3^2\right] + \sum_{i=1}^{2} dx^i dx^i + \frac{N\alpha'}{4}\sum_{i=1}^{3}(\hat{\theta}^i - A^i)^2\,. \tag{5.1}$$

Note that we can set $K = 1$ without loss of generality, since K can be rescaled by shifting the dilaton Φ by a constant. Non-extremal solutions may have a regular event horizon which is our object of interest. A solution has a regular event horizon at $r = r_h$ if ν has a simple zero there and all the other functions are finite and differentiable. Since the system is symmetric under a shift of r, we can also set $r_h = 0$. Then, such solutions will have the following Taylor expansion close to $r = 0$

$$\begin{aligned}
\nu &= K\frac{e^{2\Phi_h}}{R_h^3}r + \mathcal{O}(r^2)\,,\\
w &= w_h + \frac{e^{-2\Phi_h}}{2KR_h}(w_h^2-1)(4\kappa + (-3+4R_h^2)w_h + w_h^3)r + \mathcal{O}(r^2)\,,\\
\Phi &= \Phi_h - \frac{e^{-2\Phi_h}}{8KR_h^3}\left(16R_h^6 + 3R_h^2(w_h^2-1)^2 + (w_h^3 - 3w_h + 4\kappa)^2\right)r + \mathcal{O}(r^2)\,,\\
R &= R_h + \frac{e^{-2\Phi_h}}{8KR_h^2}\left(16R_h^4 - 4R_h^2(w_h^2-1)^2 - (w_h^3 - 3w_h + 4\kappa)^2\right)r + \mathcal{O}(r^2)\,. \qquad (5.2)
\end{aligned}$$

Here Φ_h, R_h and w_h, the value of the dilaton, the radius and w at the horizon, are free parameters. Again, we will numerically integrate (3.9)–(3.13) towards large r using (5.2) as initial conditions. Note that the value of κ is not constrained to be $\frac{1}{2}$ as in the case of the extremal globally regular solutions. Also the set of black hole solutions is three-dimensional and thus has one dimension more than the set of globally regular solutions. The extra parameter basically determines the radius of the event horizon. In order to simplify the analysis, we will set the value of the dilaton at the horizon to be $\Phi_h = 0$. Choosing a different value would simply amount to a rescaling of the solutions and not affect their qualitative structure.

In the case of $(w_h, \kappa) = (0, 0)$, we can find the solution analytically. It is given by

$$(w,\kappa) = (0,0)\,, \qquad R = \frac{1}{2}\,, \qquad \Phi = \Phi_0 - Zr\,, \qquad \nu = \frac{4}{Z^2} - K\frac{4}{Z}e^{2\Phi_0 - 2Zr}\,. \qquad (5.3)$$

Note that, setting $K = 0$, we recover the extremal solution (4.13). For $K \neq 0$, the (t, r) part of the 10d metric in the euclidean case corresponds to the "cigar" [32].

In the following, we are going to consider the case $\kappa = 1/2$ only, since this is the value corresponding to the Chern-Simons coupling k being equal to $N/2$. One can then see that for $w_h^2 > 1$ the function w diverges. Thus, we can restrict our attention to $w_h \in [-1, 1]$. For each set of values $\{\Phi_h, R_h, w_h\}$, we will obtain a black hole solution defined either on a finite interval or an infinite one. In this respect the value of R_h is important. If $R_h \gtrapprox (1/2)\sqrt{1 - w_h^2}$, then the solution extends to infinity and R is asymptotic to $\sqrt{r}$ as $r \to \infty$. On the other hand, if $R_h < (1/2)\sqrt{1 - w_h^2}$, then R goes to zero at some finite value of r.

5.2 Hawking temperature

The ten-dimensional string frame metric reads

$$ds^2_{str,10} = -\nu dt^2 + N\alpha'\left[\frac{1}{\nu}dr^2 + R^2 d\Omega_3^2\right] + \sum_{i=1}^{2} dx^i dx^i + \frac{N\alpha'}{4}\sum_{i=1}^{3}(\hat{\theta}^i - A^i)^2\,. \qquad (5.4)$$

Then, the (t, r) part of the metric continued to the euclidean region is

$$ds_2^2 = \nu d\tau^2 + \frac{N\alpha'}{\nu} dr^2 \,.$$

Close to the horizon $r = 0$, we have $\nu \sim \nu' r$, where $\nu' = (K/R_h^3)e^{2\Phi_h}$, and the metric is approximately $ds^2 = (\nu' r d\tau^2 + (N\alpha')/(\nu' r)dr^2)$. Introducing $\rho = \sqrt{(4N\alpha' r/\nu')}$ and $\theta = (\nu'/\sqrt{4N\alpha'})\tau$, the metric becomes $ds^2 = \rho^2 d\theta^2 + d\rho^2$. Thus, in order to have a regular metric, θ should be periodic with period 2π, which implies that τ has period $\beta = (4\pi/\nu')\sqrt{N\alpha'}$. The string frame metric is asymptotically flat, and the temperature at infinity will be given by

$$T = \frac{\beta^{-1}}{\sqrt{\nu(\infty)}} = \frac{K\, e^{2\Phi_h}}{4\pi R_h^3 \sqrt{N\alpha' \nu(\infty)}} \,, \tag{5.5}$$

which takes into account the normalization of ν at infinity. Note that T is invariant under a rescaling of r, which implies that the temperature does not actually depend on $\nu(\infty)$. Furthermore, T is invariant under $K \to e^{-2C} K \,, \Phi \to \Phi + C$. Therefore, the temperature will depend on three parameters only, $T = T(w_h, R_h, \kappa)$.

Again, we will rectrict our attention to the case $\kappa = 1/2$, which physically corresponds to the Chern-Simons parameter k being equal to $N/2$. We found numerically that as $R_h \to \infty$ the temperature decreases and is asymptotic to $T_c = 1/(2\pi\sqrt{N\alpha'})$, the Hagedorn temperature of the little string theory. Conversely, as the black hole radius decreases the temperature increases. In particular, for $w_h = \pm 1$, the temperature diverges in the limit $R_h \to 0$. This is the same behaviour found in [8], where the little string theory dual to four-dimensional $\mathcal{N} = 1$ Super Yang-Mills was studied.

Note also that the Hawking temperature of the black-hole solution (5.3), corresponding to $\kappa = 0$, is given by $T = 1/(\pi\sqrt{N\alpha'}) = 2T_c$.

In summary, black holes exist for any value of the temperature higher than the Hagedorn temperature T_c.

6. Free energy

Once we have obtained both the extremal and non-extremal generalizations of the regular BPS solution dual to three-dimensional super Chern-Simons theory with $k = \frac{N}{2}$, we are ready to study their contribution to the thermodynamics. In particular, we will compute the free energy of these solutions.

The ten-dimensional euclidean metric in the Einstein frame, with periodic time $\tau \in [0, \beta]$, reads

$$ds^2 = e^{-\Phi/2}\left(\nu d\tau^2 + N\alpha'\left[\frac{1}{\nu}dr^2 + R^2 d\Omega_3^2\right] + \sum_{i=1}^{2} dx^i dx^i + \frac{N\alpha'}{4}\sum_{i=1}^{3}(\hat{\theta}^i - A^i)^2\right). \tag{6.1}$$

The free energy F is defined by $I = \beta F$, where I is the euclidean ten-dimensional action, which consists of both a volume and a surface term

$$I = \frac{1}{4}\int_\Omega d^{10}x\sqrt{-g}\left(-R + \frac{1}{2}(\partial\Phi)^2 + \frac{1}{12}e^{-\Phi}H^2\right) - \frac{1}{2}\int_\Sigma \mathcal{K} d\Sigma \equiv I_{\text{vol}} + I_{\text{surf}} \,. \tag{6.2}$$

The volume integral is taken over a ten-dimensional volume Ω bounded by a nine-dimensional boundary Σ, which we take to be a hypersurface at constant r. The value of this constant will eventually be taken to infinity. $\mathcal{K}$ is the extrinsic curvature of the boundary, given by

$$\mathcal{K} = \nabla_\mu N^\mu = \frac{1}{\sqrt{g}}\partial_\mu\left(\sqrt{g}N^\mu\right),$$

where N^μ is the unit normal to Σ.

Then, by (6.1), $N^\mu = \sqrt{\nu/(N\alpha')}e^{\Phi/4}\delta^\mu_r$, and the metric induced on the boundary is

$$ds_b^2 = e^{-\Phi/2}\left(\nu d\tau^2 + N\alpha' R^2 d\Omega_3^2 + \sum_{i=1}^{2} dx^i dx^i + \frac{N\alpha'}{4}\sum_{i=1}^{3}(\hat\theta^i - A^i)^2\right)$$

which implies that $d\Sigma = (1/64)\sqrt{\nu}(N\alpha')^3 e^{-9\Phi/4}R^3 \sin\theta_1 \sin\theta_4 d\tau dx_1 dx_2 d\theta_1 \dots d\theta_6$.

The on-shell value of the volume term of the euclidean action, I_{vol}, will reduce to the integral of a total derivative, and can thus be expressed in terms of surface integrals. Using the equations of motion, which is most easily done in the string frame, we find that

$$\begin{aligned} I_{\rm vol} &= -\frac{1}{8}\int_\Omega d^{10}x \frac{\sin\theta_1\sin\theta_4}{64}(N\alpha')^{7/2}\partial_r\left(R^3 e^{-2\Phi}\frac{\nu}{\alpha' N}\partial_r\Phi\right) \\ &= \lim_{r\to\infty}\frac{1}{2}\pi^4 L^2 (N\alpha')^{5/2}\beta\left(-R^3 e^{-2\Phi}\nu\Phi'\right), \end{aligned} \tag{6.3}$$

where

$$L^2 = \int dx^1 dx^2 .$$

Note also that the lower integration limit, $r = 0$, makes no contribution, since it either corresponds to the origin of the coordinate system for regular solutions, where $R = 0$, or to the horizon in the case of black holes, where $\nu = 0$.

Let us now turn to the surface term, $I_{\rm surf}$. The extrinsic curvature is given by

$$\mathcal{K} = \frac{e^{5\Phi/2}}{R^3}\partial_r\left(R^3\sqrt{\frac{\nu}{N\alpha'}}e^{-9\Phi/4}\right).$$

Then

$$\begin{aligned} I_{\rm surf} &= -\frac{1}{2}\int_\Sigma \mathcal{K} d\Sigma = -\frac{1}{2}\int_\Sigma d^9\tilde{x}\frac{\sin\theta_1\sin\theta_4}{64}(N\alpha')^{5/2}\sqrt{\nu}e^{\Phi/4}\partial_r\left(R^3\sqrt{\nu}e^{-9\Phi/4}\right) \\ &= -\frac{1}{2}\int_\Sigma d^9\tilde{x}\frac{\sin\theta_1\sin\theta_4}{64}(N\alpha')^{5/2}\left(3R^2R'\nu e^{-2\Phi} + R^3\nu' e^{-2\Phi}/2 - \frac{9}{4}R^3\nu e^{-2\Phi}\Phi'\right) \\ &= -\lim_{r\to\infty} 2\pi^4 L^2 (N\alpha')^{5/2}\beta\left(3R^2R'\nu e^{-2\Phi} - \frac{9}{4}R^3\nu e^{-2\Phi}\Phi' + \frac{K}{2}\right), \end{aligned} \tag{6.4}$$

where we used the e.o.m. for ν, namely $\nu' = K(e^{2\Phi})/(R^3)$. Putting everything together, we find that

$$I = -2\pi^4 L^2 (N\alpha')^{5/2}\beta \lim_{r\to\infty}\left(\nu(R^3e^{-2\Phi})' + \frac{K}{2}\right). \tag{6.5}$$

Therefore, the on-shell value of the action is expressed in terms of the asymptotic values of the various fields at infinity, which we analyzed in section 4.3.

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6.1 The regularized action

The above expression for the free energy is actually divergent since the dilaton grows linearly as $r \to \infty$. Hence, it needs to be regularized. The way to do it is to subtract the value of the action for a reference background, and the natural choice is the regular BPS solution. The BPS metric is given by (6.1) with $R = R_{BPS}$, $\Phi = \Phi_{BPS}$ and with $\nu = 1$.

In order for the regularization procedure to be well-defined, the temperature of the black hole solution should be matched with the temperature of the BPS solution. To this end, we will assume that the coordinate τ has the same period β for both solutions, but we will modify the BPS metric by a constant factor ν_{BPS} in the following way

$$\begin{aligned} ds^2 &= e^{-\Phi_{BPS}/2}\Bigg(\nu_{BPS} d\tau^2 + N\alpha' \left[dr^2 + R_{BPS}^2 d\Omega_3^2\right] + \\ &\qquad + \sum_{i=1}^{2} dx^i dx^i + \frac{N\alpha'}{4}\sum_{i=1}^{3}\left(\hat{\theta}^i - A^i_{BPS}\right)^2\Bigg). \end{aligned} \tag{6.6}$$

As a result, the effective temperature of the BPS solution is given by $\beta_{\text{eff}} = \beta\sqrt{\nu_{BPS}}$.

Repeating the calculation above, we find that

$$\begin{aligned} I_{\text{vol}}(BPS) &= -\frac{1}{8}\int d^{10}x \frac{\sin\theta_1 \sin\theta_4}{64}(N\alpha')^{5/2}\partial_r\left(R^3 e^{-2\Phi}\sqrt{\nu_{BPS}}\Phi'\right)_{BPS} \\ &= -\frac{1}{2}\pi^4 L^2 (N\alpha')^{5/2}\beta \lim_{r\to\infty}\left(R^3 e^{-2\Phi}\sqrt{\nu_{BPS}}\Phi'\right)_{BPS}. \end{aligned} \tag{6.7}$$

Since the unit normal to the boundary is now $N^\mu = 1/(\sqrt{N\alpha'})e^{\Phi/4}\delta^\mu_r$, we find that

$$I_{\text{surf}}(BPS) = -2\pi^4 L^2 (N\alpha')^{5/2}\beta \lim_{r\to\infty}\sqrt{\nu_{BPS}}\left(3R^2R'e^{-2\Phi} - \frac{9}{4}R^3\Phi' e^{-2\Phi}\right)_{BPS}. \tag{6.8}$$

Finally, the regularized action $I_{reg} \equiv I - I_{BPS}$ reads

$$I_{\text{reg}} = -2\pi^4 L^2 (N\alpha')^{5/2}\beta \lim_{r\to\infty}\left(\nu(R^3e^{-2\Phi})' - \sqrt{\nu_{BPS}}\,(R^3e^{-2\Phi})'_{BPS} - \frac{K}{2}\right). \tag{6.9}$$

The free energy is then defined by

$$F \equiv \frac{I_{\text{reg}}}{\beta} = -2\pi^4 L^2 (N\alpha')^{5/2} \lim_{r\to\infty}\left(\nu(R^3e^{-2\Phi})' - \sqrt{\nu_{BPS}}(R^3e^{-2\Phi})'_{BPS} + \frac{K}{2}\right). \tag{6.10}$$

Again, in order to take this limit in a sensible way, we need to impose proper matching conditions at the boundary Σ [33]. First of all, the geometries induced on Σ must be the same in both backgrounds. Since $\Sigma \cong S^1 \times S^3 \times T^2 \times S^3_{\text{transverse}}$, the geometries will be the same if the following conditions are satisfied on Σ

$$e^{-\Phi/2}\nu = e^{-\Phi_{BPS}/2}\nu_{BPS}\,, \qquad e^{-\Phi/2}R^2 = e^{-\Phi_{BPS}/2}R^2_{BPS}\,, \qquad \Phi = \Phi_{BPS}\,, \tag{6.11}$$

and w converges to w_{BPS} sufficiently fast.

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6.2 Energy and entropy

In [33], Hawking and Horowitz showed that for stationary spacetimes admitting foliations by spacelike hypersurfaces Σ_t, the regularized free energy, obtained from the action as we did above, is related to the energy via the usual thermodynamic equation

$$F = E - TS\,, \tag{6.12}$$

where $T = \beta^{-1}$, S is the entropy, and E is the conserved ADM energy defined by

$$E = -\frac{1}{2}\int_{S_t^\infty} \sqrt{|g_{00}|}\,\left({}^8K - {}^8K_0\right) dS_t^\infty\,. \tag{6.13}$$

The integration is carried out over the 8-dimensional boundary of the 9-dimensional hypersurface Σ_t and ${}^8K$ and ${}^8K_0$ are the extrinsic curvatures of S_t^∞ in the geometry under study and in the reference background geometry respectively. The two 8-dimensional geometries on S_t^∞ must be the same, and it is also assumed that the g_{00} components are equal at S_t^∞. Finally, it is also required that the matter fields at the boundary agree at least up to a sufficiently high order [33]. As in [8], the results of this analysis can be applied to our case.

Let us use (6.13) to calculate the energy of our solutions. The metric induced on the constant time hypersurface Σ_t reads

$$ds_t^2 = e^{-\Phi/2}\left(N\alpha'\left[\frac{1}{\nu}dr^2 + R^2 d\Omega_3^2\right] + \sum_{i=1}^{2} dx^i dx^i + \frac{N\alpha'}{4}\sum_{i=1}^{3}(\hat{\theta}^i - A^i)^2\right).$$

Conversely the BPS reference solution will induce the following metric

$$ds_{t,BPS}^2 = e^{-\Phi_{BPS}/2}\left(N\alpha'\left[dr^2 + R_{BPS}^2 d\Omega_3^2\right] + \sum_{i=1}^{2} dx^i dx^i + \frac{N\alpha'}{4}\sum_{i=1}^{3}(\hat{\theta}^i - A^i_{BPS})^2\right).$$

The boundary S_t^∞ of Σ_t is defined by the hypersurface at constant r in the limit where r goes to infinity. The boundary is then given topologically by the product of two three-spheres, namely the three-sphere wrapped by the NS-branes and $S^3_{\text{trasnverse}}$, and the two-torus T^2 with coordinates x_1 and x_2.

The geometries induced on S_t^∞ will be the same if and only if

$$e^{-\Phi/2}R^2 = e^{-\Phi_{BPS}/2}R_{BPS}^2\,, \qquad \Phi = \Phi_{BPS}\,, \tag{6.14}$$

and w converges to w_{BPS} sufficiently fast. The g_{00} components of the two backgrounds agree if

$$e^{-\Phi/2}\nu = e^{-\Phi_{BPS}/2}\nu_{BPS}\,.$$

Note that these conditions are the same as (6.11) required in the evaluation of the regularized action. The unit normal to S_t^∞ is given by $n^k = \sqrt{\nu/(N\alpha')}e^{\Phi/4}\delta_r^k$, so that

$${}^8K = \frac{\sqrt{\nu}}{R^3}e^{+9\Phi/4}\partial_r\left(\frac{1}{\sqrt{N\alpha'}}R^3 e^{-2\Phi}\right).$$

Conversely, $n^k_{BPS} = 1/(\sqrt{N\alpha'})e^{\Phi_{BPS}/4}\delta^k_r$, and ${}^8K_0 = 1/R^3 e^{+9\Phi_{BPS}/4}\partial_r(1/(\sqrt{N\alpha'})R^3 \times e^{-2\Phi_{BPS}})$. Finally

$$E = -2\pi^4 L^2 (N\alpha')^{5/2} \lim_{r\to\infty} \left(\nu \left(R^3 e^{-2\Phi}\right)' - \sqrt{\nu_{BPS}}\,\left(R^3_{BPS} e^{-2\Phi_{BPS}}\right)'\right) . \tag{6.15}$$

This reproduces exactly the first term in (6.10), which agrees with the general thermodynamic relation (6.12), and yields the following expression for the entropy of the solutions

$$S = 4\pi^5 R_h^3 L^2 (N\alpha')^3 e^{-2\Phi_h} = \pi^4 L^2 (N\alpha')^{5/2}\,\beta K\,, \tag{6.16}$$

where we used the expression for the Hawking temperature, (5.5), and set $\nu(\infty) = 1$. We see that the entropy is proportional to the geometrical area of the horizon. Note also that although both the energy and the action are invariant under a translation of r, they both get a factor e^{-2C} under $\Phi \to \Phi + C$, $K \to e^{-2C}K$.

It is important to notice that for generic values of K and Φ_h a black hole solution will have $\nu(\infty) \neq 1$. Thus, in order to achieve proper normalization at infinity and keep the value of Φ_h fixed, we need to fine-tune K. Recall that the value of the dilaton at the horizon is related to the Yang-Mills coupling constant. Hence, to compare two different black hole solutions in a physically meaningful way, we have to make sure that they have the same value of Φ_h. The fine-tuning proceeds as follows. First, we use the scaling symmetry under $r \to e^{2d}r$ to set $\nu(\infty) = 1$ by taking $d = (1/4)\ln \nu(\infty)$. This changes Φ_h to $\Phi_h - d$ and leaves K invariant. Then, we use the symmetry $\Phi \to \Phi + C$, $K \to e^{-2C}K$, to set Φ to a prescribed value. This last step has no effect on $\nu(\infty)$.

Numerical analysis shows that as the temperature decreases, namely as R_h becomes larger, the black hole entropy actually increases. This is a signal of thermodynamic instability of the system above T_c.

We can now calculate the energy and free energy of a general solution using the expressions found above. Let us take a non-BPS solution and set $r_\infty = 0$ in (4.22). The regular BPS solutions make up a two-parameter family, the parameters being Φ_0 and $r_\star$, which accounts for the symmetry under translations of r. These two parameters together with ν_{BPS} will be fine-tuned so that the matching conditions (6.11) are satisfied at the boundary Σ. However, the functions w and w_{BPS} will not match exactly, unless the boundary is strictly at infinity, in which case both functions vanish. The discrepancy $\Delta w = w - w_{BPS}$ should tend to zero fast enough, otherwise the energy will be infinite. A direct calculation shows that a polynomial fall-off, $\gamma \neq 0$, is not enough for the energy to be finite. On the other hand, if the parameter γ is vanishing $\Delta w \sim e^{-2r}$ and the energy is actually finite.

6.3 Solutions with finite energy

Let us carry out the explicit computation of the energy for solutions with $\gamma = 0$. The asymptotics for large r are given by

$$R = \sqrt{2r} + \sqrt{2}\mathcal{P} r^{3/4} e^{-2r}\left(1 + \frac{1}{r} + \cdots\right), \qquad \nu = 1 - \frac{K}{2^{5/2}\, r^{3/4}} e^{-2r+2\Phi_\infty} + \cdots$$

$$\Phi = \Phi_\infty - r + \frac{3}{8}\log r + \frac{3}{2}\mathcal{P} r^{1/4} e^{-2r}\left(1 + \frac{1}{2r} + \cdots\right), \tag{6.17}$$

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where we set $r_\infty = 0$ and $\mu = 1$. The asymptotics of the BPS solution are given by

$$\begin{aligned} R_{BPS} &= \sqrt{2(r+r_*)} + \cdots \,, \qquad \Phi_{BPS} = \Phi_* - (r+r_*) + \frac{3}{8}\log(r+r_*) + \cdots \\ \nu_{BPS} &= \text{const}\,. \end{aligned} \tag{6.18}$$

We need to evaluate the limit (6.15) under the matching conditions (6.11), which are equivalent to

$$\nu = \nu_{BPS}\,, \qquad e^{-2\Phi}R^3 = e^{-2\Phi_{BPS}}R^3_{BPS}\,, \qquad R = R_{BPS}\,, \tag{6.19}$$

By the first of the above conditions, the limit becomes

$$\lim_{r\to\infty} \sqrt{\nu}\left(\sqrt{\nu}(R^3 e^{-2\Phi})' - (R^3 e^{-2\Phi})'_{BPS}\right)\,. \tag{6.20}$$

Since

$$\begin{aligned} R^3 e^{-2\Phi} &= 2\sqrt{2}r^{3/4}e^{-2\Phi_\infty+2r} + 3\sqrt{2}\mathcal{P}e^{-2\Phi_\infty} + \cdots \\ R^3_{BPS}e^{-2\Phi_{BPS}} &= 2\sqrt{2}(r+r_*)^{3/4}e^{-2\Phi_*+2(r+r_*)} + \cdots\,, \end{aligned} \tag{6.21}$$

we then obtain

$$\begin{aligned} &\lim_{r\to\infty}\sqrt{\nu}\left\{\left(1-\frac{K}{2^{7/2}r^{3/4}}e^{-2r+2\Phi_\infty}\right)\left(4\sqrt{2}r^{3/4}e^{-2\Phi_\infty+2r}+\frac{3\sqrt{2}}{2r^{1/4}}e^{-2\Phi_\infty+2r}\right)-\right. \\ &\qquad\left. -\left(4\sqrt{2}(r+r_*)^{3/4}e^{-2\Phi_*+2(r+r_*)}+\frac{3\sqrt{2}}{2(r+r_*)^{1/4}}e^{-2\Phi_*+2(r+r_*)}\right)\right\} = \\ &= \lim_{r\to\infty}\sqrt{\nu}\left\{4\sqrt{2}r^{3/4}e^{-2\Phi_\infty+2r} - 4\sqrt{2}(r+r_*)^{3/4}e^{-2\Phi_*+2(r+r_*)} + \right. \\ &\qquad\left. + \frac{3\sqrt{2}}{2r^{1/4}}e^{-2\Phi_\infty+2r} - \frac{3\sqrt{2}}{2(r+r_*)^{1/4}}e^{-2\Phi_*+2(r+r_*)}\right\} - \frac{K}{2}\,. \end{aligned} \tag{6.22}$$

By the second condition in (6.19) and (6.21) we find that

$$4\sqrt{2}(r+r_*)^{3/4}e^{-2\Phi_*+2(r+r_*)} = 4\sqrt{2}r^{3/4}e^{-2\Phi_\infty+2r} + 6\sqrt{2}\mathcal{P}e^{-2\Phi_\infty}\,.$$

Furthermore, the third condition in (6.19)

$$\sqrt{2r}(1+\mathcal{P}r^{1/4}e^{-2r}) = \sqrt{2(r+r_*)}$$

yields

$$r_* = 2\mathcal{P}r^{5/4}e^{-2r}\,.$$

Then, the limit becomes

$$\begin{aligned} &\lim_{r\to\infty}\sqrt{\nu}\left\{\frac{3}{\sqrt{2}}e^{-2\Phi_\infty+2r}\left(\frac{1}{r^{1/4}}-\frac{r^{3/4}}{r+r_*}\right)\right\} - 6\sqrt{2}\mathcal{P}e^{-2\Phi_\infty} - \frac{K}{2} = \\ &= \frac{6}{\sqrt{2}}\mathcal{P}e^{-2\Phi_\infty} - 6\sqrt{2}\mathcal{P}e^{-2\Phi_\infty} - \frac{K}{2} = -3\sqrt{2}\mathcal{P}e^{-2\Phi_\infty} - \frac{K}{2}\,. \end{aligned} \tag{6.23}$$

Finally, by (6.15), the ADM energy of a non-BPS solution with $\gamma = 0$ will be given by

$$E = 2\pi^4 L^2 (N\alpha')^{5/2} \left(3\sqrt{2}\mathcal{P} e^{-2\Phi_\infty} + \frac{K}{2} \right) . \tag{6.24}$$

Note that the energy is invariant under constant shifts of r and is therefore independent of r_∞. Conversely, under $\Phi \to \Phi + C$, $\mathcal{P}$ is invariant while $K \to e^{-2C}K$, and the energy picks up an overall factor e^{-2C}.

Let us calculate the action via

$$I = \beta E - S .$$

For the globally regular solutions, the entropy vanishes and $K = 0$ that gives

$$I_{\text{global}} = 2\pi^4 L^2 (N\alpha')^{5/2} \left(3\sqrt{2}\, \beta\, \mathcal{P}\, e^{-2\Phi_\infty} \right) . \tag{6.25}$$

For black holes, the entropy S and the term proportional to K in the expression for the energy cancel out and we find

$$I_{BH} = 2\pi^4 L^2 (N\alpha')^3 (3\sqrt{2}) \frac{4\pi}{K} \mathcal{P} R_h^3 e^{-2\Phi_h - 2\Phi_\infty} , \tag{6.26}$$

where we used the expression for the Hawking temperature (5.5) and normalized the solution to set $\nu(\infty) = 1$. Again, under $\Phi \to \Phi + C$, $\mathcal{P}$ and R_h remain invariant while $K \to e^{-2C}K$, so that the action picks up the overall factor e^{-2C}.

6.4 Globally regular solutions with finite energy

The numerical analysis of the system (4.17)–(4.20) shows that there are actually no finite energy globally regular solutions besides the regular BPS one.

Figure 1 is a plot of γ as a function of b, which parametrizes the family of globally regular solutions and belongs to the interval $[0, 1[$. We see that γ has a single zero, whose position is compatible with the supersymmetric solution, which corresponds to $b = \frac{1}{3}$ (Fig. 2).

6.5 Black hole solutions with finite energy

In the case of black-holes, a similar analysis shows that there is in fact a single finite energy solution for each value of the temperature above $T_c = 1/(2\pi\sqrt{N\alpha'})$.

In figures 3, 4 and 6, we show a plot of γ as a function of the parameter $w_h \in]-1, 1[$ for a fixed value of R_h and a constant value of the temperature T.

The existence of finite energy black holes may allow a Hawking-Page transition above T_c [34]. Below T_c, there are no black holes, the BPS solution with periodic euclidean time dominates the path integral and the field theory is confining. On the contrary, above T_c, if a given black hole had a negative action, then its contribution to the path integral would be more relevant than the BPS one and the dual field theory would undergo a transition to a deconfined phase. By (6.26), in order to detect such a transition one has to determine $\mathcal{P}$. This is quite difficult since $\mathcal{P}$ is a coefficient in front of subleading terms which vanish exponentially and we have not been able to estimate it with accuracy.

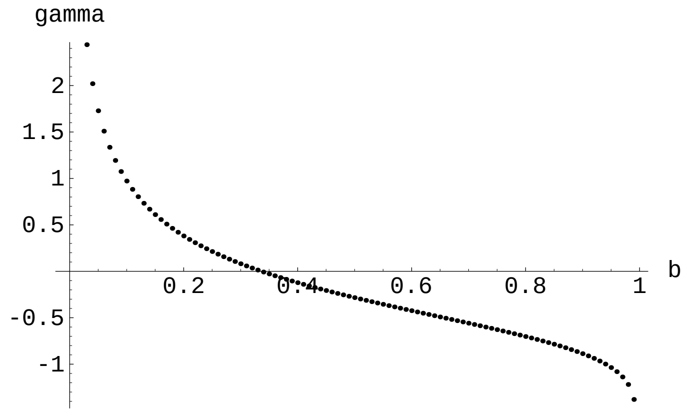

Figure 1: Globally regular solutions:plot of γ as a function of b.

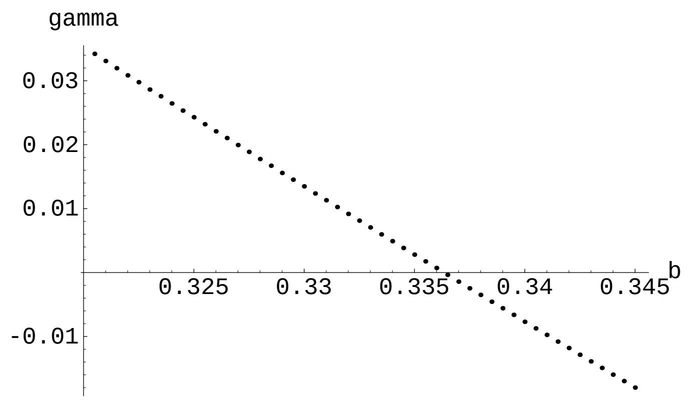

Figure 2: Plot of γ expanded around the node.

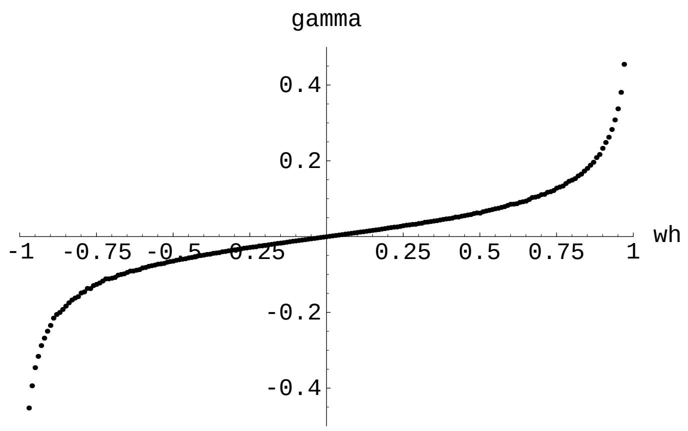

Figure 3: Plot of γ as a function of w_h for $R_h = 50$ and T/T_c around 1.00.

However, as we remarked in section 6.2 , the fact that the black hole entropy (6.16) decreases as the temperature increases tells us that the would be high temperature phase is actually unstable. It would have negative specific heat. Therefore, such a transition

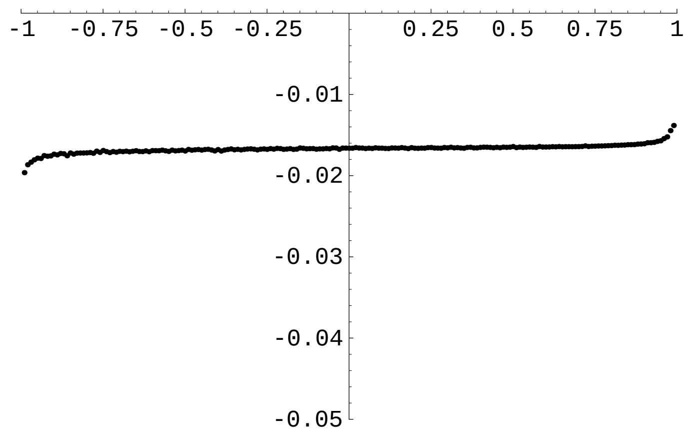

Figure 4: Plot of γ as a function of w_h for $R_h = 10$ and T/T_c around 1.01.

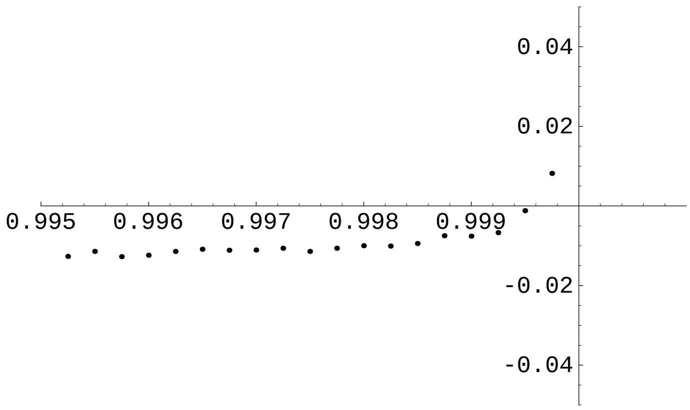

Figure 5: Still $R_h = 10$ and T/T_c around 1.01. Expanded version of Fig.4 to show the node which is very close to $w_h = 1.0$.

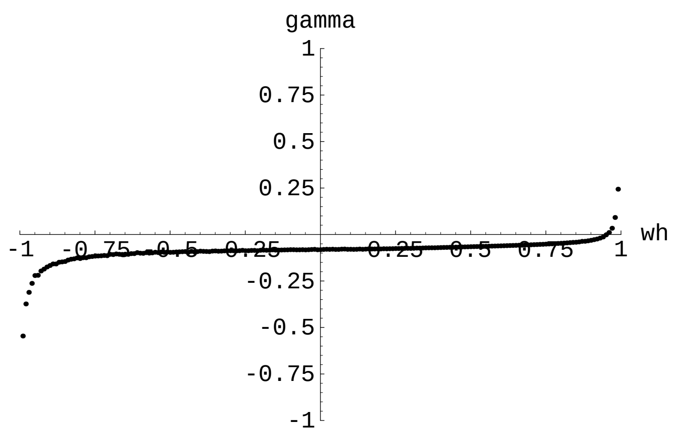

Figure 6: Plot of γ as a function of w_h for $R_h = 2$ and T/T_c around 1.12.

between the stable low-temperature confining phase and the unstable high-temperature deconfining phase may not take place. This was first pointed out in [8], where a similar system of NS5-branes wrapping a shrinking S^2, which is dual to 4d $\mathcal{N}=1$ SYM [9], was studied in detail. In this case, the authors actually determined that the black hole action becomes negative above T_c. A similar analysis was carried out earlier in [36], where it was shown that a system of NS5-branes wrapping a two-sphere in a resolved conifold had negative specific heat.

The underlying little string theory is believed to have an exponential growth in the number of states around T_c and is actually thermodynamically unstable above this temperature [35]. The results of [36, 8] and the present work confirm the presence of this instability.

This thermodynamic instability is thought to be due to a Gregory-Laflamme instability [37, 38] of the underlying system of black NS5-branes. Based on results found in the context of the AdS_4 Reissner-Nordstrom solution, Gubser and Mitra conjectured that *for a black brane with translational symmetry, a Gregory-Laflamme instability exists precisely when the brane is thermodynamically unstable* [39, 40]. Further arguments were given in [41]. Along these lines, Rangamani [42] argued that the instability of the little string theory found in [35] is actually due to the presence of a *threshold unstable mode* [41] that survives the non-decoupling limit of the non-extremal NS5-branes. It would be interesting to see whether such a mode exists in the present context.

7. Conclusions

We have studied both extremal and non-extremal generalizations of the regular supersymmetric solution dual to 3d U(N) $\mathcal{N}=1$ Super Yang-Mills with Chern-Simons coupling $k=N/2$, originally found by Chamseddine and Volkov in the context of $5d$ $N=4$ gauged supergravity. We rederived both the singular BPS solution (4.10) and the regular one (4.11) and (4.12).

We have found an interesting analytical non-supersymmetric solution corresponding to a system of NS5-branes wrapping a constant radius S^3 (4.13). This geometry factorizes into $S^3 \times S^3$ times a non-compact part with linear dilaton (4.14) and (4.15). It has a non-trivial H field (4.16) and admits a non-extremal generalization with a Hawking temperature $T=2T_c$, where T_c is the Hagedorn temperature of the little string theory (5.3).

We have also found that there are no finite energy extremal globally regular solutions except the BPS one. Furthermore, all black hole solutions have a temperature which is larger than T_c and their entropy decreases as the temperature increases. This indicates that the system is thermodynamically unstable above T_c as was also found in [36, 8].

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